Welcome to the fifth module of the MOOC Algorithmic Information Dynamics. As probably, you know by now my name is Narsis and today I will talk to you about dynamical systems.

At the end of the 17th century, Leibniz (1646-1716) and Newton (1643-1727), independently one from the other, invented a brilliant mathematical tool: infinitesimal calculus or differential and integral calculus. This is an incredibly efficient crystal ball to predict the future, provided the system in question is governed by a differential equation. Using it , Poincaré's work on celestial mechanics (Poincaré 1899), and specifically in a 270-page, prize-winning, and initially flawed paper (Poincaré 1890) start the qualitative theory of dynamical systems. The methods developed therein laid the basis for the local and global analysis of nonlinear differential equations, including the use of first-return (Poincaré) maps, stability theory for fixed points and periodic orbits, stable and unstable manifolds, and the Poincaré recurrence theorem. ~~More strikingly, using the example of a periodically perturbed pendulum, Poincaré showed that mechanical systems with n≥2 degrees of freedom may not be integrable, due to the presence of homoclinic orbits.~~

In this module, the idea is to make you a very generic introduction to dynamical systems . we talk about how systems typically only occupy a small subset of the overall space as they cycle through some set of states. We will be talking about attractors and the fundamental role they play within dynamics of a system. We will discuss chaotic and complex regime consisting of multiple attractors and equilibria very briefly. The concept of Boolean network as a discrete dynamical system will be introduced and we learn how we can analyse them.

Ok let see what we mean with dynamical system. Within science and mathematics, dynamics is the study of how things change with respect to time, as opposed to describing things simply in terms of their static properties. The patterns we observe all around us in how the state of things change overtime is an alternative ways through which we can describe the phenomena we see in our world. A dynamical system is a set of possible states, together with a rule that determines the present state in terms of past states. As examples for dynamical systems, you can think of any system that is evolving in time. For example, the pendulum, or whether evolution, or the evolution of population of bacteria or any kind of season that evolves through time.

A dynamical system have two parts State space and function, and we describe such a system by them. So lets see what they are. Dynamical system is study of the thing, which are changing overtime! Those things are states and a state space is a model used within dynamic systems to capture this change in a system’s state overtime.

Formally, State space is the set of all possible states of a dynamical system. Each state of the system corresponds to a unique point in the state space. For example, the state of an idealized pendulum is uniquely defined by its angle and angular velocity, so the state space is the set of all possible pairs "(angle, velocity)", which form the cylinder.

In general, any abstract set could be a state space of some dynamical system.

It could be finite, consisting of just a few points or consisting of an infinite number of points forming a smooth manifold, as usually the case in ordinary differential equations and mappings. Such a state space is often called a phase space. A state space could be infinite-dimensional, as in partial differential equations and delay differential equations. In symbolic dynamics, it is a Cantor set, which is zero-dimensional. Moreover, the second part of dynamical system, Function tells us, given the current state, what the state of the system will be in the next instant of time.

For investigating dynamical systems, it is necessary to specify some characteristics that provide a subdivision into special classes of dynamical systems. Specific methods are available for some of these classes, thus such a classification can help to simplify the analysis.

A dynamical system is deterministic If the present state can be determined uniquely from the past states (no randomness is allowed). Stochastic models possess some inherent randomness. Chaotic model is a deterministic model with a behaviour that cannot be entirely predicted. They are predictable in the very short term, but appears random for longer periods.

An important characteristic of a dynamical system is whether it is continuous or discrete. In a discrete system, the state variables change only at a countable number of points in time. These points in time are the ones at which the event occurs/change in state. In a continuous dynamical system, the state variables change in a continuous way, and not abruptly from one state to another (infinite number of states). When the reals are acting, the system is called a continuous dynamical system, and when the integers are acting, the system is called a discrete dynamical system.

Continuous systems are given by differential equations whereas discrete dynamical systems (often called maps) are specified by difference equations. Let’s start by discrete, We denote time by k or n, and the system can be solved by iteration calculation called iterative maps. Iterative maps give us less information but are much simpler and better suited to dealing with very many entities, where feedback is important. Typical example is annual progress of a bank account. If the initial deposit is 100000 euros and annual interest is 3%, then we can describe the system by



8: In a continuous system, the time interval between our measurements is negligibly small making it appear as one long continuum and this is done through the language of calculus and using differential equation or a set of them.

For example, vertical throw is described by initial conditions h(0), v(0) and equations



where ***h*** is height and ***v*** is velocity of a body.

Calculus and differential equations have formed a key part of the language of modern science since the days of Newton and Leibniz. Even though an analytical treatment of dynamical systems is usually very complicated, obtaining a numerical solution is (often) straightforward. Solving differential equations numerically can be done by a number of schemes. The techniques for solving differential equations based on numerical approximations were developed before programmable computers existed. During WorldWarII, it was common to find rooms of people (usually women) working on mechanical calculators to numerically solve systems of differential equations for military calculations.

The easiest way is by the 1st order Euler’s Method which uses the idea of local linearity or linear approximation, where we use small tangent lines over a short distance to approximate the solution to an initial-value problem. If we zoom in small enough, every curve looks like a straight line, and therefore, the tangent Line is a great way for us to calculate what is happening over a period. Today we have many numerical methods to solve differential equations. And there is no "best way" or "best method", since the method to be chosen heavily depends on the problem (stiff or non stiff, equation or system, smooth or non smooth right hand side, etc.). For a "general" or "standard", a 4th order Runge-Kutta is good, it's easy to add an error estimator to it with no or little additional cost. A predictor-corrector Adams-method also does the job, they are also quite popular.

Differential equations are great for few elements they give us lots of information but they also become very complicated very quickly. Where as differential equations are central to modern science iterative maps are central to the study of nonlinear systems and their dynamics as they allow us to take the output to the previous state of the system and feed it back into the next iteration, thus making them well designed to capture the feedback characteristic of nonlinear systems.

11: Another important classification of dynamical system is based on linearity. Let start by defining a Nonlinear System. Nonlinear System is a set of (one or more)nonlinear equations. Nonlinear equations are equations where the unknown quantity that we want to solve for appears in a nonlinear fashion. For example, if the quantity in question is a function y(t),then terms such as y^2or sin y would be nonlinear. More precisely, a nonlinear equation is one where a linear combination of solutions is not a new solution. In a linear system, function that is describing the system behaviour must satisfy two basic properties

* + additivity
  + homogeneity



Which one is linear? g(x) = 3x; g(y) = 3y or f(x) = x2; f(y) = y2?

* additivity g(x+y) = 3x + 3y = g(x) + g(y)
* homogeneity 5 \* g(x) = 5\* 3x = 15x = g(5x)

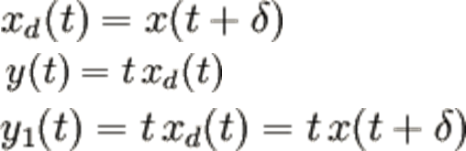


If a system of ordinary differential equations do not depend on the independent variable, it called an Autonomous system. If the independent variable is time, we call it time-invariant system. In an autonomous system If the input signal x(t) produces an output y(t) then any time shifted input, x(t + δ), results in a time-shifted output y(t + δ). Consider these two systems,

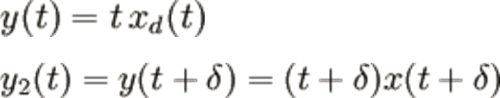
System A: System B:

Are they ***Autonomous?***

We examine system A and u can do the same for system B. lets Start with a delay of the input



Now we delay the output by δ



Clearly , therefore the system is not time-invariant or is nonautonomous.

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The first step in the analysis of a dynamic system is to derive its model. Models may assume different forms, depending on the particular system and the circumstances. Mathematical model of a dynamic system can often be expressed as a system of differential (difference in the case of discrete-time systems) equations. The response of dynamic system to an input may be obtained if these differential equations are solved. The differential equations can be obtained by utilizing physical laws governing a particular system, for example, Newtons laws for mechanical systems, Kirchhoffs laws for electrical systems, etc. In obtaining a model, we must make a compromise between the simplicity of the model and the accuracy of results of the analysis. In this lecture we learn how we can do that by study two real world dynamical systems but before that we need to learn and review some concepts related to state space.

As we learned in pervious lecture, at any given time, a dynamical system has a state given by a vector that can be represented as a point in a geometrical manifold. What future states follow from the current state is described by the evolution rule of the dynamical system. We also defined the set of all states of a system of ordinary differential equations as phase space.

If we plot the solution of equations of motion in a phase space then we have a phase curve. The phase curve is dependent on initial state if we plot single- or multiple phase curves corresponding to different initial conditions in the same phase plane, then we have a phase portrait. To fully understand the behaviour of a dynamical system, we need to know how a system move from one position to another and this is described by trajectories through state space. A trajectory or path is set of positions in state space through which system might pass successively.

Now lets look at a real world dynamical system, A biological system containing two species – predators (foxes) and prey (rabbits). There are literally hundreds of examples predator prey relations in an ecosystem, predation is a biological interaction where an organism that hunting feeds on its prey. There is a continuous tussle between predators and their prey and an inverse relationship between the number of predators and prey.

If we take fox – rabbit as an example then

Rabbits, left on their own, will reproduce with velocity dr/dt= 100 r, foxes, without rabbits, will starve and their population will decline with velocity dw/dt= -50f. When brought into the same environment, foxes will catch and eat rabbits. Loss to the rabbit population will be proportional to number of foxes f and number of rabbits r.

The predator–prey model was initially proposed by Lotka in the theory of autocatalytic chemical reactions in 1910 and Volterra developed his model independently from Lotka and used it to explain d'Ancona's observation his son in law regarding increasing the fauna population during world war I in Adriatic. Lotka–Volterra model are a pair of first-order nonlinear differential equations, frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations you see here:

One of the motion you might encounter and we would like to model here is periodic motion, for example, the motion of the planets around the sun is periodic. This type of periodic motion is very predictable and we can predict far out into the future and way back into the past when eclipses happen. In these systems small disturbances are often rectified and do not increase to alter the systems trajectory very much in the long run. The change in traffic lights or particle motion, under a force, linear in the displacement is are also example of periodic motion. A differential equation of motion, usually identified as some physical law and applying definitions of physical quantities, is used to set up an equation for the problem. Solving the differential equation will lead to a general solution with arbitrary constants, which results in to a family of solutions. A particular solution can be obtained by setting the initial values, which fixes the values of the constants.

21: now if we solve the equation for Particle motion, under a force then we can see that the motion is simple harmonic in each of the two dimensions. Both oscillations have the same frequency but (in general) different amplitudes.

The amplitudes A, B, & the phases α, β are determined by initial conditions. So different initial conditions gives different solution.

22: If we want to plot phase portrait for this system we need to find all of its paths. An equation for path is obtained by eliminating t in the solution. And trick for this system is defining new parameter δ ≡ α – β then we can see that except for special cases, the general path is an ellipse like!

Therefore, the phase portrait for this system is a family of ellipses, each of which is a separate phase curve for different initial conditions. Studying such diagrams gives insight into the physics of the particle motion. As homework, you can try to create phase portrait of this oscillator in Mathematica.

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In this lecture, I will talk about dynamical system analysis. We can solve equations for particular starting point(s), but this is often not enough to enable us to understand the system. Therefore, we use complementary analysis focused on finding equilibrium states (or stationary/critical/fixed points) where system remains unchanged over time. Analysis determines how the system behaves over time, in particular investigating future behaviour of the system given any current state ie the long-term behaviour of the system. Also we try to classify these states/points as stable/unstable by investigating the behaviour of the system near them.

We learn how to find fixed points of a system and classify them as stable/unstable for 1 (space) dimensional systems and 2 (and higher) dimensional systems but Will NOT talk about proofs of many of the stability theorems , many of special cases focus will be on how to use them to analyse your systems . These lectures are a primer for understanding the power and advantages of using algorithmic complexity based tools for analysis dynamical systems.

So lets start with something you can test right now, try to balance a coin on a table! In how many ways you can do that, I cannot do it in more than 2 ways but there is 3d one, Head, tails and edge. So let assume that you are one of the expert ones and could balance a coin at its edge, is it stable?

… No! Small movement away (perturbation) means that coin will end up in one of 2 different equilibria: Heads/ Tails.

How about head and tail? Yes they are both stable as perturbation does not result in a change in state.

25: Similarly, think of a ball at rest in a dark landscape. It’s either on top of a hill or at the bottom of a valley. To find out which, push it (perturb it), and see if it comes back. (What about flat bits?)

26: So a set towards which a dynamical system evolves over time called ***attractor*** . It can be a point, a curve or more complicated structure. A ***perturbation*** is a small change in a physical system, most often in a physical system at equilibrium that is disturbed from the outside.

Each attractor has a basin of attraction, which contains all the initial conditions that will generate trajectories joining asymptotically this attractor.

27: When studying a nonlinear dynamical system, if we are only interested in the long-time behaviours, we will only study the attractors of the system and determine their basin of attractions. one of the “simplest” attractors are the point: it is then a fixed point, i.e. The particular points of the phase space verifying dX/dt(x∗)= 0. The corresponding solution of the dynamical system does not depend on time. It is a stationary state.

28: So what are the fixed points of this system dx/dt = 6x(1 – x) ?

If we solve 6x(1-x) = 0 we can find 2 fixed points:

Are they stable?

To test that we perturb the points and see what happens but to perturb them we must have a model of the system.

We can use difference equation: x(t+h) = x(t) +h dx/dt from various different initial x’s

29: If we perturb each fixed point a bit and see if the output of system will change or not we can see that X=0 unstable and x=1 stable.

There are a general rules regarding the stability of one-dimensional dynamical system:

If a0 is a fixed point:

If f’(a0) > 0, a0 is unstable

if f’(a0) < 0, a0 is stable

if|f’(a0) = 0, inconclusive.

where f’(a0) means the derivative of f, df/dx evaluated at a0. if f’(a0) = 0 need to use higher derivatives or other methods and can be semi-stable from above or below or periodic …

**31:** so by now we know a fixed point is a special point of the dynamical system which does not change in time. It is also called an equilibrium, steady-state, or singular point of the system.

If a system is defined by an equation dx/dt = f(x), then the fixed point 𝑥 ̃ can be found by examining of condition f(𝑥 ̃)=0. We do not need to know analytic solution of x(t).

A stable fixed point: for all starting values x0 near the 𝑥 ̃ the system converges to the 𝑥 ̃ as t→∞.

A marginally stable fixed point: for all starting values x0 near 𝑥 ̃, the system stays near 𝑥 ̃ but does not converge to 𝑥 ̃. An unstable fixed point: for starting values x0 very near 𝑥 ̃ the system moves far away from 𝑥 ̃.

Now let’s finish the lecture by looking at another biological model , bacteria Growth Model.

We leave a nutritive solution and some bacteria in a dish. Let b be relative rate at which the bacteria reproduce and p be relative rate at which they die. Then the population is growing at the rate r = b−p. If there are x bacteria in the dish then the rate at which the number of bacteria is increasing is (b − p)x, that is, dx/dt = rx. Solution of this equation for x(0)=x0 is 

However, is this mode realistic?

No, The model is not realistic, because bacteria population goes to the infinity for r>0. But in reality as the number of bacteria rises, they produce more toxic products. So instead of constant relative perish rate p, we will assume relative perish rate dependent on their number px. So the number of bacteria increases by bx and their number decreases by px2 and new differential equation will be



How many-fixed point has this model?

* To be able to find a fixed point, we have to set the right-hand side of the differential equation to zero. There are two possible solutions, we have two fixed points:



Lets see what they mean and check their stability

First fixed point means, there are no bacteria, none can be born, none can die. However, after small contamination (perturbation) which is smaller than b/p, the number of bacteria will increase by dx/dt = bx-px2>0 and will never return to the zero state. First fixed point is unstable.

And Second point = b/p, at this population level, bacteria are being born at a rate b2/p and are dying at the same rate, so birth and death rates are exactly in balance.

If the number of bacteria would be slightly increased, then dx/dt = bx-px2<0 and would return to equilibrium. If the number of bacteria would be slightly decreased, then dx/dt = bx-px2>0 and would return to equilibrium. Small perturbations away from = b/p will self-correct back to b/p. Therefore, second fixed point is stable. How does the graphic solution in Mathematica look like?

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In this section, I will talk about higher-dimensional systems. In higher-dimensional systems, movement of trajectories can exhibit a wider range of dynamical behaviour. Fixed points still exist, but can be more interesting depending on how trajectories approach or repel from the equilibrium point for example system could spiral in to a stable point. Other types of stability exist such as saddle-nodes, and importantly cyclic/periodic behaviour: limit cycles. So More interesting, but more difficult to analyse…

Similar to one-dimensional system I will cover how to find fixed points and classifying fixed points for these systems

38: Suppose we have the following system:



First we need to learn how we can find fixed points and then examine stability of fixed points and finally we will examine the phase plane and trajectories.

To find fixed points as before we need to solve dx/dt = 0 and dy/dt = 0 to get fixed points (x0, y0).

So let’s look at a Predator-prey system:



so again, **(0,0)** is fixed point. Other fixed point at **(80, 12).**

To examine behaviour at/near fixed points we can examine their change against time or in phase plane. View in phase-space (or phase-plane) where x plotted against y rather than against time gives more information about system.

As you see, the system can show cyclic behaviour that is fixed but system is not at a fixed point: complications of higher dimensions.

Before going forward we need to define Jacobian matrix.

**Jacobian matrix** is the matrix of all first-order partial derivatives of a vector- or scalar-valued function with respect to another vector. This matrix is frequently being marked as J, Df or A. Jacobian matrix is used to classify fixed points of higher order linear dynamical system

Suppose x0 =(x0, y0)T is a fixed point. Define the Jacobian:

Find eigenvalues and eigenvectors of J evaluated at the fixed point:

1. If eigenvalues have negative real parts, x0 asymptotically stable
2. If at least one has positive real part, x0 unstable
3. If eigenvalues are pure imaginary, stable or unstable

Various behaviours depending on the eigenvalues (ei) and eigenvectors can be seen. Here you can see some examples for 2d linear dynamical system. In general, points attracted along negative eigenvalues and repelled by positive. Axes of attraction are eigenvectors.

If we go back to our example:



Then we have fx = 0.6 – 0.05y, fy = 0.05x, gx = 0.005y, gy = 0.005x – 0.4

For J(0,0) eigenvalues 0.6 and –0.4 and eigenvectors (1,0) and (0,1). Unstable (a saddle point) with main axes coordinate axes. For J(80,12) eigenvalues are pure imaginary.

* For a dynamical system to be stable:
  + The real parts of all eigenvalues must be negative.
  + All eigenvalues lie in the left half complex plane.

A dynamical system is *Underdamped* if there is spiral fixed point (some complex eigenvalue) and we say it is *Overdamped* if exhibit nodal behaviour (all eigenvalues real) and *Critically damped* at the boundary between.

We can classify dynamical systems using trace and determinant of the Jacobian matrix. Is there any classification for linear 3D systems using Eigen values?

Lets look at more examples for 2d systems

Consider the following system:





The Jacobean matrix will be

And Eigen values are λ1= -1, λ2= -4 and Eigen vectors will be

There is a stable fixed point, the attracting node (sink).



Now if we change the system to

And Eigen values are λ1= 1 , λ2= 4 which means it is an unstable fixed point, the repelling node

In contrast, of attractor, a repeller is a point of state space away from which system will tend when in surrounding region.

The third system has another type of unstable fixed point called saddle point.

For non-linear equations, behaviour near the fixed points will be ‘almost like’ the behaviour of a linear system depending how ‘almost linear’ it is. Behaviour gets less linear-like the further away trajectories get from the fixed point.

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In the early 1960's Edward Lorenz, a MIT meteorologist was working on developing a System to simplify the convection rolls in upper atmosphere for long-range weather prediction (5+ days).

However, the weather is complicated! A theoretical simplification was necessary. In 1963 derived a three dimensional system in efforts to model long range predictions for the weather

Using this system ran heading into "sensitivity to initial conditions". In the process, he sketched the outlines of one of the first recognized chaotic attractors. And he came to the conclusion that model equation are inaccurate in their representation of some aspect of the weather OR the model may be accurate but there is some anomalous property of the equations that makes our prediction difficult.

The Lorenz systems describes the motion of a fluid between two layers at different temperature. Specifically, the fluid is heated uniformly from below and cooled uniformly from above. By rising the temperature difference between the two surfaces, we observe initially a linear temperature gradient, and then the formation of Rayleigh-Benard convection cells. After convection, turbulent regime is also observed

The Lorenz attractor defines a 3-dimensional trajectory by the differential equations:



σ, r, b are parameters

When calculating this model, Lorenz encountered a strange phenomenon. After entering slightly different input values for two successive attempts he obtained completely different outputs.

This effect was later named a butterfly effect; the "Butterfly Effect" is the propensity of a system to be sensitive to initial conditions. Such systems over time become unpredictable, this idea gave rise to the notion of a butterfly flapping it's wings in one area of the world, causing a tornado or some such weather event to occur in another remote area of the world

The following graphs show time dependence of functions x(t) and z(t) in the Lorenz attractor for the recommended parameter values, while **blue** curves are related to initial conditions x(0)=1; y(0)=1; z(0)=**10** and **red** curves are related to initial conditions x(0)=1; y(0)=1; z(0)=**10.01**

Very slight change of initial condition results in large change in solution of the function.



If we calculate the Jacobian of the system at S1 then we have 

And Eigenvalues of J are



If you do the same for the other two points then for the recommended parameter values all three fixed points are **unstable**, because they all include an eigenvalue with positive real part.

A chaotic system is roughly defined by sensitivity to initial conditions: infinitesimal differences in the initial conditions of the system result in large differences in behaviour. Chaotic systems do not usually go out of control, but stay within bounded operating conditions.

Chaos provides a balance between flexibility and stability, adaptiveness and dependability. It lives on the edge between order and randomness. How do you think algorithmic complexity can be used to analysis this type of system? What if you do not have equations describing the system and you only observe the output of system close to its attractors?

As you saw dynamical system can have different type of attractors. If the system evolves towards a single state and remains there, we call it fixed point. An example is damped pendulum or a sphere at the bottom of a spherical bowl. It will be Periodic or quasiperiodic attractor when the system evolves towards a limit cycle. An example is undamped pendulum or a planet orbiting around the Sun. If the system is very sensitive to initial conditions and we are not able to simply predict its behaviour, we call it Chaotic attractor:. An example is the Lorenz attractor and finally the system has strange attractor if it is also very sensitive to initial conditions and we are not able to simply predict its behaviour, but in this case, the system has the same properties like fractals. In another words, the strange attractor represents a fractal. An example is the Mandelbrot set.

Fractal is a geometric shape, A fractal is a non-regular geometric shape that has the same degree of non-regularity on all scales. Fractals can be thought of as never-ending patterns. It has a Hausdorff dimension of its border higher than the topological dimension of the border. A**n example is a Cantor set:** where the original line is divided into three parts while the middle part is erased. The same procedure is applied to newly created lines etc. if we are repeating this procedure to the infinity, we obtain an infinite number of points with topological dimension 0. The set contains n=2 copies of itself reduced to 1/3 of the original dimension (k=3). Hausdorff dimension is log(2)/log(3)=0.6309…, which is greater than 0.

~~One other concepts that you might hear a lot about in the field of dynamical system is Bifurcation. The detailed discussion is out of the scope of this course and we just review the definition.~~

~~Bifurcation is a qualitative change of phase portraitt in the area of attraction which can be achieved by change of driving parameter when passing through the critical value.   
The critical points (equilibrium solutions) usually depend on the value of parameters of the model. As parameters steadily increases or decreases, it often happens that at a certain value of a, called a bifurcation point, critical points come together, or separate, and equilibrium solutions may either be lost or gained.~~

~~We distinguish two principal types of bifurcations:~~

~~Global bifurcation: its effects are not limited by the neighbourhood of a point or cycle in the phase space. It can not be detected by analysis of fixed point stability.~~

~~Local bifurcation: its effects are limited by the neighbourhood of a point or cycle in the phase space. Fixed points may appear or disappear due to the parameter change, they change their stability, or even break apart into periodic points. Such bifurcation can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets.~~

Up to now we learned dynamical systems live in **phase space** and develop in **time, following their dynamical law. We saw examples of** continuous dynamical system and learned how to analysis them. In the next two lectures, we will do the same for discrete dynamical system.

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We talked that dynamical systems are about the evolution of some quantities over time. This evolution can occur smoothly over time or in discrete time steps. Here, we introduce dynamical systems where the state of the system evolves in discrete time steps, i.e., discrete dynamical systems. A discrete dynamical system is like taking a snapshot of the system at a sequence of times. The snapshots could occur once a year, once every month, or even irregularly, such as once any one coughs. For example, the pendulum is a 2-dimensional **continuous** **dynamical system**. A Cellular Automaton or Boolean network are a **discrete dynamical system**.

When we take these snapshots, the idea is that we are recording whatever variable determine the state of the system: our chosen state variables that evolve through the state space . To complete the description of the dynamical system, we need to specify a rule that determines, given an initial snapshot, what the resulting sequence of future snapshots must be.

You already know about CAs lets study Boolean network as another class of discrete dynamical system. A Boolean network is - a directed graph *G*(*V,E*) that characterized by the number of nodes : *N and*  the number of inputs per node: *k*

Boolean models are discrete (in state and time) and deterministic. Each node can assume one of two states: on („1“) or off („0“). When we do not have enough information for more detailed description or beacouse of Increasing complexity and computational effort for more specific models, we will use boolean models.

Boolean networks have always a finite number of possible states: 2*N*  and, therefore, a finite number of state transitions , .

For example in the graph shown here , there are theoretically possible state transition.

The dynamics of a boolean network are described by set of rules:

„if input value/s at time *t* is/are...., then output value at *t*+1 is....“

For example in the following graph

A(t)

B(t+1)

So we can define different rules for same network:

For example Rule 0 is saying independent of input, output is always 0 and in rule 4 is always 1 while rule 2 is negation and therefore the state of node B at time t+1 equals  **logical complement of**  the state of node A at time t.

70 : Before going forward, we need to define some terminology:

State of a network is row listing the present value of all N nodes (0 or 1). The system have finite number of states (2N) and so as system passes along a sequence of states from an arbitrary initial state, it must eventually re-enter a state previously passed 🡪 a cycle

*If a cycle* contains merely one state, it is a singleton attractor; otherwise, it is an attractor cycle.

The surrounding region in state space such that all trajectories starting in that region end up in the attractor is called basin of attraction. Basins of attraction is a mathematical object that can be computed and shown as graphs for small networks. In biology, Multiple attractors explain how the same genetic regulatory network can maintain different stable patterns of gene activation,the celltypes in multi-cellularorganisms. And if perturbed can cause the dynamics to jump to alternative attractors.

Now if you consider a chain which state of node A is fixed, the system using either rules 1 or 2 will reach to a steady state after N-1 time steps:

A(t) 🡪 B(t+1)

B(t+1) 🡪 C(t+2)

C(t+2) 🡪 D(t+3)

Let’s look at another example, a Ring with 2 nodes:

If we start from 00 or 11 , and follow the rule 1 then we stay at same state , they are attractor of the system but if we start either from 01 or 10 , we find a cycle of length 2 . Using rule 2 we enter a Cycle of length 4 independent of initial conditions. A Boolean Net has 2N possible states, but many fewer basins of attraction

In 1969, Stuart Kauffman proposed using Random Boolean Networks (RBNs) as an abstract model of gene regulatory networks. Where Each vertex represents a gene and “on” state of a vertex indicates that the gene is expressed. An edge from one vertex to another implies the the former gene regulates the later and “0” and “1” values on edges indicate presence/ absence of activating/repressing proteins. Boolean functions assigned to vertices represent rules of regulatory interactions between genes

If we consider a 3 genes system with following rules

X(*t*+1) = X(*t*) and Y(*t*)

Y(*t*+1) = X(*t*) or Y(*t*)

Z(*t*+1) = X(*t*) or (not Y(*t*) and Z(*t*))

Then we can calculate all possible transient states for the given network. We can demonstrate all attractors and their basin of attraction using a graph called, state transition graph or STG.

So we can see that the number of accessible states is finite, 2^N and Cyclic trajectories are possible. - Not every state must be approachable from every other state. The successor state is unique; the predecessor state is not unique.

If the rules for updating states are unknown, we need to select rules randomly. So suppose that Boolean functions are assigned to RBN vertices so that they evaluate to 0 with probability p and evaluate to 1 with probability 1-p. For example, p = 0.5 means that Boolean functions are assigned independently and uniformly at random from the set of 16 Boolean functions of 2 variables.

An *NK* automaton is an autonomous random network of *N* Boolean logic elements. Each element has *K* inputs and one output. The signals at inputs and outputs take binary (0 or 1) values. The Boolean elements of the network and the connections between elements are chosen in a random manner. There are no external inputs to the network. The number of elements *N* is assumed large. An automaton operates in discrete time. The set of the output signals of the Boolean elements at a given moment of time characterizes a current state of an automaton. During an automaton operation, the sequence of states converges to a cyclic attractor. The states of an attractor can be considered as a "program" of an automaton operation. The number of attractors *M* and the typical attractor length *L* are important characteristics of *NK* automata.

With K connections, there is 22K Boolean input functions; these Nets are free of external inputs.

Once, connections and rules are selected, they remain constant and the time evolution is deterministic.

What is the relationship between the average connectedness of genes and the ability of organisms to evolve? Has fortunate evolutionary history selected only nets of highly ordered circuits that alone insure metabolic stability; Or are stability and epigenesist, even in nets of randomly interconnected regulatory circuits, to be expected as the probable consequence of as yet unknown mathematical laws? Are living things more akin to precisely programmed automata selected by evolution, or to randomly assembled automata…?

Given N bulbs and K connections behavior is

1) Chaotic: If K is large, the bulbs keep twinkling chaotically

2) Frozen or periodic: If K is small (K = 1), some flip on and off, most soon stop

3) Complex: If K is around 2, complex patterns appear, in which twinkling islands of stability develop, changing shape at their borders.

A network that is frozen either solid or chaotic cannot transmit information and thus cannot adapt.

Gene regulatory networks of living cells are believed to exhibit phase transition behavior, on the border between the frozen and chaotic phases. Kaufman has shown that if k = 2 and p = 0.5, then the statistical features of RBNs match the characteristics of living cells

* + number of attractors ≈ number of cell types
  + length of attractors ≈ cell cycle time

When we study a system, our motivation is usually a search for causal relations. Although in everyday life we frequently make causal statements, such as “I couldn’t get up on time this morning because I was up late last night”, in general we cannot “see” causal relations but can only infer their existence. Our current systems theory, *including all that is taken from physics or physical science,* deals exclusively with *simple systems* or *mechanisms*. While Complex and simple systems are disjoint categories. von neuman thought that a critical level of system size would trigger the onset of complexity but Complexity is more a function of system qualities rather than size. Complex systems require that all aspects of them be encoded in order to be more completely understood. This is not possible only using traditional parameter dependent modelling. The world of simple mechanisms is a surrogate world created by traditional science. The real world is complex and a new view is needed.

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